Notes.

(a) You may freely use any result proved in class unless you have been asked to prove the same. Use your judgement. All other steps must be justified.

(b) We use $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers.

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1. [6 + 12 = 18 Points]
   (i) State Thales' theorem and the affine version of Pappus' theorem.
   (ii) Do any one of the following.
   (a) Give a proof of Pappus' theorem assuming Thales' theorem.
   (b) State and prove the projective version of Pappus' theorem assuming the affine version.

2. [24 Points] Let $\mathcal{C}, \mathcal{C}'$ be two circles in the plane intersecting orthogonally at two distinct points. Let $\mathcal{D}$ be a line tangent to both $\mathcal{C}, \mathcal{C}'$. In each of the following cases, describe the effect of inversion I with pole O (and with some power) on these curves, how they get positioned with respect to each other and how they intersect or touch each other. You should also draw representative pictures to illustrate your answers.
   (i) $O \notin \mathcal{C} \cup \mathcal{C}' \cup \mathcal{D}$.
   (ii) $O \in \mathcal{C}$, and $O \notin \mathcal{C}' \cup \mathcal{D}$.
   (iii) $O \in \mathcal{C} \cap \mathcal{D}$.
   (iv) $O \in \mathcal{C} \cap \mathcal{C}'$.

3. [18 Points] Let $\mathcal{E}$ be a Euclidean affine space of dimension 3 over $\mathbb{R}$. Let α be rotation by angle π with respect to some axis in $\mathcal{E}$. Let β be reflection about some plane in $\mathcal{E}$.
   (i) Can $\beta \circ \alpha$ be a translation in $\mathcal{E}$? Justify your answer.
   (ii) Describe what kind of isometry $\alpha \circ \beta$ is.
   (iii) Describe what kind of isometry $\alpha \circ \beta \circ \alpha^{-1}$ is.

4. [12 Points] The faces of a convex polyhedron $\mathcal{P}$ in 3-dimensional Euclidean space are given to be regular pentagons and regular hexagons, with all the edges having the same side-length. If each vertex lies in exactly 3 faces, find the number of pentagonal faces in $\mathcal{P}$.

5. [12 Points] Let k be a field. Find all projective linear maps (homographies) $\mathbb{P}^1_k \to \mathbb{P}^1_k$ that send ∞ to 0 and 0 to 1. You may express the maps in homogenous coordinates or in matrix form (inducing an element of $\mathrm{PGL}_2(k)$).

6. [16 Points] Define the cross ratio $[a, b, c, d]$ of four distinct collinear points a, b, c, d in $\mathbb{P}^n_k$. If $a_1, a_2, \ldots, a_m$ are distinct collinear points in $\mathbb{P}^n_k$ with $m \geq 4$, then prove that

$$\prod_{i=3}^{i=m-1} [a_1, a_2, a_i, a_{i+1}] = [a_1, a_2, a_3, a_m].$$